

A SPECTRAL SEQUENCE FOR LAGRANGIAN FLOER HOMOLOGY

REZA REZAZADEGAN

ABSTRACT. We prove the existence of a spectral sequence for Lagrangian Floer homology which converges to the Floer homology of the image of a Lagrangian submanifold under multiple fibred Dehn twists. The E_1 term of the sequence is given by the hypercube of “resolutions” of the Dehn twists involved. The proof relies on the exact triangle for fibered Dehn twists due to Wehrheim and Woodward.

As applications we obtain a spectral sequences from Khovanov homology to symplectic Khovanov homology. Also when a 3-manifold M is given by gluing two handlebodies by a surface diffeomorphism ϕ , we obtain a spectral sequence converging to the Heegaard-Floer homology of M , whose E_1 term is a hypercube obtained from different ways of resolving the Dehn twists in ϕ . This latter sequence generalizes the spectral sequence of branched double covers to general closed 3-manifolds (i.e. those which are not branched double covers of links) however its E_2 term is not a 3-manifold invariant. This gives upper bounds on the rank of HF-hat of closed 3-manifolds.

1. INTRODUCTION

It is a well-known fact of homological algebra that whenever we have an iterated mapping cone of chain maps between complexes C_i then there is a spectral sequence whose E_1 term is given by the direct sum of the cohomologies of the C_i and converges to the cohomology of the iterated mapping cone. In this paper we apply this principle to mapping cones arising from fibered Dehn twists along coisotropic submanifolds of symplectic manifolds. Such Dehn twists arise naturally in a variety of contexts and give rise to actions of the braid and mapping class groups by symplectomorphisms and therefore to link and 3-manifold invariants.

Assume we have a coisotropic submanifold C in a symplectic manifold M which fibers over a (symplectic) manifold B with sphere fibers. For example, C can be a Lagrangian sphere. Denote by Δ_M the diagonal in $M^- \times M$ and let τ_C^\pm denote positive/negative fibered Dehn twist along C . (See Section 2.) Wehrheim and Woodward [25] (generalizing a result of Seidel [21]) prove that for any two Lagrangian submanifolds L, L' of M satisfying suitable monotonicity conditions, the Lagrangian Floer chain complex $CF(L, \tau_C L')$ is quasi-isomorphic to the mapping cone of the map

$$CF(L, C^t, C, L')[\dim_{\mathbb{C}} B] \xrightarrow{\mu} CF(L, \Delta_M, L') \cong CF(L, L')$$

where the left hand side is quilted Floer homology and the map μ is given by counting pseudoholomorphic quilted triangles (quilted pairs of pants) as in Figure 1. Here $C^t C$ and Δ_M can be regarded as zero and one “resolutions” of the fibered Dehn twist τ_C . It follows from the invariance and duality properties of Floer homology that $CF(L, \tau_C^- L')$ is quasi-isomorphic to the cone of

$$\left(CF(L, \Delta_M, L')[\dim_{\mathbb{C}} B] \xrightarrow{\mu^t} CF(L, C^t, C, L') \right)[-1]$$

where μ^t is induced by the transpose of the pseudoholomorphic quilts that give μ .

Given a collection C_1, C_2, \dots, C_N of spheric coisotropic submanifolds of M (where C_i fibers over a manifold B_i) and a vector of signs $\mathcal{E} = (\epsilon_1, \epsilon_2, \dots, \epsilon_N)$, we combine the result of Wehrheim and Woodward with the above principle to obtain a spectral sequence converging to

$$(1) \quad HF(L, \tau_{C_N}^{\epsilon_N} \circ \tau_{C_{N-1}}^{\epsilon_{N-1}} \circ \dots \circ \tau_{C_1}^{\epsilon_1}(L'))$$

from the N -dimensional hypercube obtained by 2^N ways of resolving the τ_{C_i} . More precisely the hypercube is given as follows. If $\epsilon_i = 1$ set $C_i^0 = C_i^t C_i$ and $C_i^1 = \Delta_M$ which are both generalized correspondences from M to itself. Otherwise set $C_i^0 = \Delta_M$ and $C_i^1 = C_i^t C_i$. For $I = (I_1, I_2, \dots, I_N) \in \{0, 1\}^N$ set

$$(2) \quad CF_I = CF(L, C_N^{I_N}, C_{N-1}^{I_{N-1}}, \dots, C_1^{I_1}, L')$$

which is quilted Lagrangian Floer chain complex. (See section 4.2 for definitions.) Denote $\sigma(I) = \sum_{I_i=0} \dim_{\mathbb{C}} B_i$ and let n_- denote the number of negatives among the ϵ_i . As a chain group the hypercube is given by

$$(3) \quad CF_{\oplus} = \bigoplus_{I \in \{0,1\}^N} CF_I[\sigma(I) - n_-].$$

The maps between the adjacent vertices of the hypercube are given by counting rigid pseudoholomorphic quilted triangles. In general there are nonzero maps $\mu_{I,J}$ between nonadjacent vertices of the cube corresponding to I, J . These maps are given by counting special families of pseudo-holomorphic quilted polygons. See Section 4 for details. *In this paper we work entirely with Floer homology over $\mathbb{Z}/2$.*

Theorem 1.1. *Assuming M, L, L' and the C_i satisfy the Admissibility Assumption of Definition 4.1, there is a finite cubic filtration on the chain complex of (1). Therefore there is a spectral sequence converging to (1) whose first page is given by*

$$(4) \quad E_1 = \bigoplus_I HF(L, C_N^{I_N}, C_{N-1}^{I_{N-1}}, \dots, C_1^{I_1}, L')[\sigma(I) - n_-]$$

with d_1 being the sum of the maps $\mu_{I,J}$, between adjacent vertices, given by counting quilted pairs of pants.

See Theorem 6.1 for a more precise statement and the proof. The spectral sequence exists more generally for L, L' generalized Lagrangian submanifolds of M and is natural with respect to equivalence of Lagrangian correspondences (Prop. 6.5).

Corollary 1.2. *With the same assumptions as in Theorem 1.1, if Lagrangian Floer homology groups are \mathbb{Z}/n graded ($n = \infty$ for \mathbb{Z} grading) then*

$$(5) \quad \dim HF^i(L, \tau_{C_N}^{\epsilon_N} \circ \tau_{C_{N-1}}^{\epsilon_{N-1}} \circ \dots \circ \tau_{C_1}^{\epsilon_1} L') \leq \sum_j \sum_{\sigma(I) - n_- + j = i \pmod n} \dim HF^j(L, C_N^{I_N}, C_{N-1}^{I_{N-1}}, \dots, C_1^{I_1}, L').$$

If the Floer homology groups are not graded the inequalities still hold with for ungraded homology ($n = 0$).

In the case where the C_i are Lagrangian spheres (or equivalently B_i 's are points), we have

$$HF(L, C_N^{I_N}, C_{N-1}^{I_{N-1}}, \dots, C_1^{I_1}, L') \cong HF(L, C_{k_1}) \otimes HF(C_{k_1}, C_{k_2}) \otimes \dots \otimes HF(C_{k_{m-1}}, C_{k_m}) \otimes HF(C_{k_m}, L')$$

where $m \leq N$ and k_i 's are so that $C_{k_i}^{I_{k_i}} \neq \Delta$. Also d_1 is given by the count of pseudoholomorphic triangles either with one (for $\epsilon_i = 1$) or two (for $\epsilon_i = -1$) outgoing ends. One feature of our

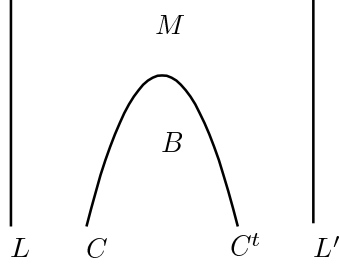


FIGURE 1. A quilted pair of pants

spectral sequence which seems to be new, is that it involves holomorphic quilts (e.g. polygons) with multiple outgoing ends (for negative twists).

A typical situation where we have composition of Dehn twists, and so iterated mapping cones, is when we have a representation of the braid group on the symplectomorphism group of a manifold e.g. in symplectic Khovanov homology of Seidel and Smith [23]. In particular we obtain a spectral sequence converging to symplectic Khovanov homology from Khovanov homology (with $\mathbb{Z}/2$ coefficients). (See section 7.1.)

1.1. Application to Heegaard-Floer homology. Fibered Dehn twists also give rise to actions of the mapping class groups of surfaces by symplectomorphisms. An important example of this kind is given by Perutz's (and Lekili's) approach to Heegaard-Floer homology. Let Σ_g be a surface of genus g . Perutz [15] assigns to each embedded circle $\gamma \subset \Sigma_g$ a Lagrangian correspondence V_γ between the symmetric products $Sym^g \Sigma_g$ and $Sym^{g-1} \Sigma_{g-1}$ (with appropriate symplectic forms). Let H, H' be two handlebodies with $\partial H = \partial H' = \Sigma_g$. Denote by T_H and $T_{H'}$ their corresponding Heegaard tori.

Theorem 1.3. *Let M be a closed oriented 3-manifold obtained by gluing H to H' by a homeomorphism ϕ of the surface Σ_g . Let ϕ be given as a composition of Dehn twists along curves in Σ , $\phi = \tau_{\gamma_N}^{\epsilon_N} \circ \dots \circ \tau_{\gamma_1}^{\epsilon_1}$. Choose a basepoint $z \in \Sigma$ away from the curves α_i, β_i and γ_i . Then there is a spectral sequence which converges to $\widehat{HF}(M)$ and whose E_1 term is given by*

$$(6) \quad \bigoplus_I HF(T_{H'}, V_{\gamma_N}^{I_N}, \dots, V_{\gamma_1}^{I_1}, T_H).$$

In the above theorem the symplectic manifolds involved are $Sym^g \Sigma_g \setminus (\{z\} \times Sym^{g-1} \Sigma_g)$ and $Sym^{g-1} \Sigma_{g-1} \setminus (\{z\} \times Sym^{g-2} \Sigma_{g-1})$ with the same Kähler forms used in Perutz construction. One also Hamiltonian isotopes the submanifolds V_{γ_i} to become balanced. The differential d_1 is again given by counting pseudoholomorphic quilted pairs of pants. Note that since $c_1(Sym^g \Sigma_g)$ is nonzero (even mod n), the above homology groups are ungraded.

Theorem 1.3 is interesting even in the case $N = 1$ where it is a reincarnation of the knot surgery exact triangle [14] in terms of the mapping class group. Each summand in (6) is the \widehat{HF} of a 3-manifold given as follows. Let Y_{γ_i} denote the cobordism between Σ_g and Σ_{g-1} in which γ_i is excised out and let $Y_{\gamma_i}^{I_i}$ be defined with the same rule as for $V_{\gamma_i}^{I_i}$ (eg. if $\epsilon_i = 1$, and $I_i = 0$ then

$Y_{\gamma_i}^{I_i} = Y_{\gamma_i} Y_{\gamma_i}^t$). Then it follows from the work in progress of Lekili and Perutz [7] that the I' 'th summand of (6) is the \widehat{HF} of the 3-manifold given by the sequence of cobordisms $H, Y_{\gamma_N}^{I_N}, \dots, Y_{\gamma_1}^{I_1}, H'$.

We compute the E_1 page in the case that M is the double cover of S^3 branched over a link L (denoted $\Sigma(K)$) and show that the E_1 page is Khovanov's hypercube for L (with coefficients in $\mathbb{Z}/2$). More precisely we have the following.

Theorem 1.4. *In Theorem 1.3 let ϕ be hyperelliptic, so its mapping cylinder is the double cover of $S^2 \times [0, 1]$ branched over a braid $b \in Br_{2g}$. Also assume $H' = H$ be a handlebody of genus g . Then the page (E_1, d_1) of the spectral sequence of Thm. 1.3 is naturally isomorphic to the Khovanov hypercube (over $\mathbb{Z}/2$) for the plat closure K of the mirror of b and its E_∞ page is $\widehat{HF}(\Sigma(K) \# S^1 \times S^2)$.*

The proof relies on work in progress by Lekili and Perutz [7]. (See 7.5 below.) The spectral sequence of Thm. 1.4 is very closely related to the spectral sequence of Ozsvath and Szabo [14]. However the (higher) maps in our hypercube are not exactly given by (higher) cobordism maps for Heegaard-Floer homology. Lipshitz, Ozsvath and Thurston [8] have been working on obtaining an explicit computation of the spectral sequence of Ozsvath and Szabo using bordered Heegaard-Floer homology. Our approach can provide an alternative, more geometric method which we hope to get back to in future.

Remark 1.5. *It may be tempting to call the E_2 page of the spectral sequence of Theorem 1.3 the “Khovanov homology of the 3-manifold M ”. However it has been shown by Watson [24] that Khovanov homology of a link K does not give an invariant of the branched double cover of K . This, together with Thm. 1.4, implies that this E_2 page is not a 3-manifold invariant. Nonetheless in a forthcoming work we study this E_2 term in more detail and give a combinatorial description for it.*

Question 1.6. *Which 3-manifolds have presentations for which the spectral sequence of Theorem 1.3 collapses at E_2 page?*

It follows from the work of Ozsvath and Szabo [14, Prop 3.3] together with Thm. 1.4 that branched double covers of quasi-alternating links have this property. It is possible that the manifolds in question are an appropriate generalization of L-spaces. (L-spaces are rational homology 3-spheres M for which $\dim_{\mathbb{Z}/2} \widehat{HF}(M) = \#H_1(M, \mathbb{Z})$).

Corollary 1.2 now gives upper bounds on the rank of $\widehat{HF}(M)$ from each presentation of M by gluing of handlebodies. One possible application of these inequalities is in deciding whether a given 3-manifold can be obtained from a given composition of (classical) Dehn twists. We end this introduction by noting that obtaining such a spectral sequence for a topological invariant, defined using Lagrangian Floer homology, can be regarded as the first step toward obtaining a combinatorial description of the invariant.

Remark 1.7. *The same arguments that are used to prove the Theorem 1.1 can be adapted to prove a hypercube in the derived Fukaya category $DF^\#(M, M)$. This means that, under the same assumptions as in Theorem 1.1, the Lagrangian correspondence $\text{graph}(\tau_{C_{N-1}}^{\epsilon_{N-1}} \circ \dots \circ \tau_{C_1}^{\epsilon_1})$ is isomorphic, in $DF^\#(M, M)$, to an element of the form $(\sum_I (C_N^{I_N}, \dots, C_1^{I_1}), D)$ where D is given by the count of quilts similar to the ones that give the differential on the hypercube. See Proposition 6.3.*

Organization. In section 2 we recall basic facts about spheric coisotropic submanifolds. In section 3 we recall some examples of coisotropic submanifolds of importance to low dimensional topology. Section 4 is the technical part of the paper in which we construct the hypercube under different

admissibility conditions on the manifolds involved. The exact triangle for fibered Dehn twists is reviewed in section 5. In section 6 we prove Theorem 1.1. Theorems 1.3 and 1.4 are proved in sections 7.2 and 7.3 respectively.

Acknowledgements. I thank Paolo Ghiggini for his lectures on Heegaard-Floer homology and for a helpful discussion. I would also like to thank Robert Lipshitz for pointing out the relevance of Watson's result and Tim Perutz for helpful correspondence. A part of this work was done when the author was visiting the Mathematical Sciences Research Institute as a research member. The author was also partly supported by QGM (Center for Quantum Geometry of Moduli Spaces).

2. PRELIMINARIES

The main objects of study in this paper are fibered Dehn twists along spherically fibered coisotropic submanifolds. Such submanifolds are closely related to symplectic Morse-Bott fibrations (also called Lefschetz-Bott fibrations) which are generalizations of symplectic Lefschetz fibrations. Let (M, ω) be a symplectic manifold and $C \subset M$ coisotropic (which means that for any $x \in C$ and any $v \in T_x M$, if $\omega(v, w) = 0$ for all $w \in T_x C$ then v is tangent to C). Because of the closedness of ω , the distribution $\ker \omega|_C$ is integrable and the resulting foliation is called the *null foliation* of C .

Definition 2.1. A coisotropic submanifold C of a symplectic manifold (M, ω) is fibered if there is a manifold \bar{M} and a fibration $\pi : C \rightarrow \bar{M}$ which is constant on the leaves of the null foliation. It is spherically fibered (or just spheric) if the fibers of π are spheres S^k and moreover the structure group of the bundle π can be reduced to $SO(k+1)$ where k is the codimension of C .

The manifold \bar{M} inherits a symplectic form ω given by

$$(7) \quad \bar{\omega}(\pi_* v, \pi_* w) = \omega(v, w).$$

Let $\iota : C \rightarrow M$ be the inclusion then $(\iota, \pi) : C \rightarrow M^- \times \bar{M}$ is a Lagrangian embedding and so we can regard C as a Lagrangian correspondence from M to \bar{M} (or equivalently $C^t = (\pi, \iota)C$ a correspondence from \bar{M} to M).

$SO(k+1)$ acts on \mathbb{C}^{k+1} with a moment map $\eta : \mathbb{C}^{k+1} \rightarrow \mathfrak{so}(k+1)^*$ whose regular fiber is S^k . The $SO(k+1)$ -bundle associated to π yields an associated \mathbb{C}^{k+1} -bundle P over \bar{M} . Let $\alpha \in \Omega^1(P, \mathfrak{so}(k+1))$ be a connection one form on P and let π_1, π_2 be the projections from P to \bar{M} and \mathbb{C}^{k+1} respectively. One has a closed 2-form on P given by

$$(8) \quad \omega' = \pi_1^* \bar{\omega} + \pi_2^* \omega_{\mathbb{C}^{k+1}} + d\langle \eta, \alpha \rangle$$

where $\langle, \rangle : \mathfrak{so}(k+1)^* \otimes \mathfrak{so}(k+1) \rightarrow \mathbb{R}$ is the pairing. This form is nondegenerate in a neighborhood P_ϵ of the zero section [5].

Consider T^*S^k as a subset of \mathbb{C}^{k+1} given by the pairs (\mathbf{x}, \mathbf{y}) such that $|\mathbf{x}| = 1, \langle \mathbf{x}, \mathbf{y} \rangle = 0$. Let $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ be monotone increasing on $[0, 1]$, send this interval to $[\pi, 2\pi]$ and be equal to π or 2π outside this interval. The *generalized Dehn twist* τ along the zero section is defined by

$$(9) \quad \tau \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \cos(\psi(|\mathbf{y}|)) \cdot I & |\mathbf{y}|^{-1} \sin(\psi(|\mathbf{y}|)) \cdot I \\ -|\mathbf{y}| \sin(\psi(|\mathbf{y}|)) \cdot I & \cos(\psi(|\mathbf{y}|)) \cdot I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

where I is the $(k+1) \times (k+1)$ identity matrix. If σ_t is the Hamiltonian flow of the length function $|\mathbf{y}|$ then $\tau(\mathbf{x}, \mathbf{y}) = \sigma_{\psi(|\mathbf{y}|)}(\mathbf{x}, \mathbf{y})$. It is easy to see that τ extends smoothly to the zero section and is the antipodal map there. One can see by direct computation that τ is a symplectomorphism for the canonical symplectic structure on T^*S^k .

Now there is a symplectomorphism $\tilde{\tau}$ of (P_e, ω') which is a generalized Dehn twist along the S^k in each \mathbb{C}^{k+1} fiber. More precisely $\tilde{\tau}$ is the time 2π map of the Hamiltonian flow of $\psi \circ \eta$. By the coisotropic neighborhood theorem [5, Thm 39.2] a neighborhood of C in M is symplectomorphic to a neighborhood of the zero section in P_e . This way $\tilde{\tau}$ induces a symplectomorphism τ_C of M called the *fibred Dehn twist* along C . The Hamiltonian isotopy class of τ_C is not changed by the action of Hamiltonian isotopies on C . However it is not known if this class is independent of the “framing” i.e. the choice of a local symplectomorphism into P_e .

3. SOME EXAMPLES OF SPHERIC COISOTROPIC SUBMANIFOLDS

Example 3.1 (following Seidel and Smith [23]). Let $\mathcal{S}_m \subset \mathfrak{sl}_{2m}(\mathbb{C})$ be the set of matrices of the form

$$(10) \quad \begin{pmatrix} y_1 & & & I \\ y_2 & & I & \\ \vdots & & \ddots & \\ y_{n-1} & & & I \\ y_n & & & 0 \end{pmatrix}$$

where I is the 2×2 identity matrix, $y_1 \in \mathfrak{sl}_2$ and $y_i \in \mathfrak{gl}_2$ for $i > 1$. \mathcal{S}_m is a transverse slice to the adjoint orbit of a nilpotent matrix of Jordan type (m, m) in \mathfrak{sl}_{2m} [23, Lemma 23]. Let Σ_{2m} denote the symmetric group on $2m$ letters and consider the map $\chi : \mathcal{S}_m \rightarrow \mathbb{C}^{2m}/\Sigma_{2m}$ which sends each matrix to its spectrum. Set

$$(11) \quad \mathcal{Y}_{m,\nu} = \chi^{-1}(\nu).$$

If ν has no repetitions then it is a regular value of χ and so $\mathcal{Y}_{m,\nu}$ is a Kähler (in fact Stein) manifold with vanishing first Chern class. Denote by Conf_{2m} the set of regular values of χ , i.e. $2m$ -tuples without multiplicities. Let $\delta : [0, 1) \rightarrow \text{Conf}_{2m}$ be any curve such that $\nu' := \lim_{t \rightarrow 1} \delta(t)$ has an element μ of multiplicity two and no more repetitions among its members. Let $\bar{\nu} \in \text{Conf}_{2m-2}$ be the result of deleting μ from ν' . It can be easily shown that the set of critical points of χ in $\mathcal{Y}_{m,\nu'}$ is symplectomorphic to $\mathcal{Y}_{m-1,\bar{\nu}}$ so we can regard $\mathcal{Y}_{m,\nu'}$ as a submanifold of $\mathcal{Y}_{m,\nu'}$.

Let $D \subset \mathbb{C}$ be a small disk containing the image of δ and such that $D \cap \text{Conf}_{2m} = D \setminus \{\delta(1)\}$. Seidel and Smith prove [23, Lemma 27] that up to a Kähler isomorphism the restriction of χ to D is of the form $\pi(x, a, b, c) = a^2 + b^2 + c^2$. Let $U \subset \mathcal{Y}_{m-1,\bar{\nu}}$ be compact. Let $L_\delta = L_\delta(U) \subset \mathcal{Y}_{m,\mu}$ be the set of points of $\mathcal{Y}_{m,\delta(t)}$ (for t close to 1) which converge to an element of $U \subset \mathcal{Y}_{m-1,\bar{\nu}} \subset \mathcal{Y}_{m,\nu'}$ under the gradient flow of $\text{Re}\pi$. L_δ is a relative vanishing cycle for the fibration $\chi|_D$. Morse lemma implies that after possibly replacing ν with $\delta(t)$ for t close to 1, the map $L_\gamma \rightarrow \mathcal{Y}_{m-1,\bar{\nu}}$ which sends a point to its limit under the gradient flow is smooth and is a S^2 -bundle over its target. So, L_δ is a spheric coisotropic submanifold of $\mathcal{Y}_{m,\nu}$ which fibers over $U \subset \mathcal{Y}_{m-1,\bar{\nu}}$. Moreover if we choose a larger compact subset U' , the restriction of the resulting $L_\delta(U')$ to U is Hamiltonian isotopic to $L_\delta(U)$.

This construction has been generalized by Manolescu [9] to the case of $\mathfrak{sl}_{n,m}$. The analogs of the above coisotropic submanifolds are $\mathbb{C}P^n$ bundles in that case.

Example 3.2 (following Perutz [15]). Let S be a Riemann surface of genus g and $\gamma \subset S$ an embedded circle. Let \bar{S} be the result of surgering γ out and attaching two discs to $S \setminus \gamma$. Let $\eta \in H^2(\text{Sym}^8 S)$ be Poincare dual to $\{pt\} \times \text{Sym}^{8-1} S$ and θ be such that $\theta - g \cdot \eta$ is Poincare dual to $\sum a_i \times b_i \times \text{Sym}^{8-2} S$

where $\{a_i, b_i\}$ is a symplectic basis for $H_1(S)$. Let $P_S \in H^2 \text{Sym}^8 S$ be a cohomology class of the form $t\eta + s\theta$ where s, t are constants. As in [17] there are symplectic forms in this class which agrees with the push-forward of the product symplectic form (on $S \times S \times \cdots \times S$) away from the big diagonal. We similarly have a cohomology class $P_{\bar{S}}$ for $\text{Sym}^{8-1} \bar{S}$ (with the same values of s and t).

Theorem 3.3 (Perutz [15], Theorem A). *If ω and $\bar{\omega}$ are symplectic forms in cohomology classes P_S and $P_{\bar{S}}$ respectively then there is a coisotropic submanifold $\iota : V_\gamma \rightarrow (\text{Sym}^8 S, \omega)$ which fibers over $(\text{Sym}^{8-1} \bar{S}, \bar{\omega})$ with circle fibers.*

As in Example 3.1, V_γ is the vanishing cycle for a Lefschetz-Bott fibration over the disk. The generic fiber of this fibration, which we denote by p , is $\text{Sym}^8 S$ and its set of critical points can be identified with $\text{Sym}^{8-1} S$. More specifically one starts with a Lefschetz fibration π over the disk in which S becomes nodal along γ and then one considers the fibration p where the $p^{-1}(z)$ for each point z is the Hilbert scheme of points on $\pi^{-1}(z)$. This Hilbert scheme for nonsingular curves (i.e. $\pi^{-1}(z)$ for $z \neq 0$) is the same as the symmetric product. See [15] for details.

Example 3.4 (following Wehrheim and Woodward [29]). Let $\Sigma_{g,n}$ denote a topological surface of genus g with n punctures and let $\mathcal{M}_{g,n}$ denote the moduli space of flat $\text{SU}(2)$ connections on $\Sigma_{g,n}$ whose holonomy around each puncture has trace zero. If n is odd then this moduli space is smooth [29, Prop. 3.3.1]. $\mathcal{M}_{g,n}$ also has a symplectic structure which goes back to Atiyah and Bott. See [2] for a more modern approach.

Let $\gamma \subset \Sigma_{g,n}$ be an embedded circle which does not bound a (punctured) disk and whose complement is connected. Then one can consider the three dimensional cobordism Y_γ between $\Sigma_{g,n}$ and $\Sigma_{g-1,n}$ in which γ is pinched to a point and then excised out. Consider $C_\gamma \subset \mathcal{M}_{g,n} \times \mathcal{M}_{g-1,n}$ consisting of pairs of connections which extend to the whole of Y_γ . Note that projection on the first factor embeds C_γ in $\mathcal{M}_{g,n}$. For two nearby punctures z_1, z_2 one can also consider the cobordism Y_{z_1, z_2} between $\Sigma_{g,n}$ and $\Sigma_{g,n-2}$ in which z_1 and z_2 merge. It gives rise to a subset $C_{z_1, z_2} \subset \mathcal{M}_{g,n}$.

Theorem 3.5 (Wehrheim-Woodward [29], Prop. 3.4.2). *C_γ is a smooth coisotropic submanifold of $\mathcal{M}_{g,n}$; it fibers over $\mathcal{M}_{g-1,n}$ with S^3 fibers. The set C_{z_1, z_2} is also a smooth coisotropic submanifold of $\mathcal{M}_{g,n}$ which fibers over $\mathcal{M}_{g,n-1}$ with S^2 fibers.*

So an elementary cobordism between surfaces gives a Lagrangian correspondence between the corresponding moduli spaces of flat connections. Because $C_\gamma \subset \mathcal{M}_{g,n}$ is coisotropic, one can consider the fibered Dehn twist τ_{C_γ} along it. A result of Wehrheim and Woodward [25] (extending results of Callahan and Seidel) shows that fibered twist along C_γ agrees up to Hamiltonian isotopy, to the diffeomorphism induced on $\mathcal{M}_{g,n}$ by the classical Dehn twist along γ .

4. THE HYPERCUBE

In this section we construct the maps between the vertices of the hypercube of (3) and show that they make (3) into a chain complex. These maps are given by the count of pseudoholomorphic quilts. The conditions that guaranty that the maps in the hypercube are well-defined and give rise to a differential turn out to be the same as the conditions that guaranty the Fukaya category $\mathcal{F}(M)$ of the symplectic manifold M is well-defined. Bubbling of pseudoholomorphic discs with boundary on a single Lagrangian or pseudoholomorphic polygons with fixed Maslov index but with unbounded energy prevent the Fukaya category from being well-defined. The latter problem is prevented by imposing the monotonicity (or exactness) condition on the Lagrangians while for the first problem there are a number of different remedies. Below we recall a few well-known situations in which $\mathcal{F}(M, \omega)$ is well-defined.

- (i) M is monotone and one includes only the Lagrangians L which are monotone, the image of $\iota : \pi_1(L) \rightarrow \pi_1(M)$ is torsion and the minimal Maslov number of L is greater than two.
- (ii) ω is exact with convex (contact type) boundary and one includes only its compact exact Lagrangian submanifolds (i.e. if $\omega = d\sigma$ then $\sigma|_L$ is exact) [22].
- (iii) M is monotone with a prequantum line bundle K with a connection form α and one includes only balanced (also called Bohr-Sommerfeld monotone) Lagrangians L (i.e. $K|_L$ has a section s for which $s^*\alpha$ is exact) with the additional condition that $\pi_2(M, L) = 0$.
- (iv) M is a (noncompact) Stein manifold of finite type and one includes only exact Lagrangian submanifolds which are invariant under the Liouville flow outside a compact subset [19, Section 4.3].

Definition 4.1 (Admissibility Condition). *A symplectic manifold M and a collection (L, C_1, \dots, C_k, L') where $L, L' \subset M$ are Lagrangian and $C_i \subset M$ are spheric coisotropics fibering over manifolds B_i are said to be admissible if they satisfy one of the above conditions (C_i as a Lagrangian submanifold of $M^- \times B_i$). For (i) and (iii) the C_i and L, L' are assumed to have the same monotonicity constant.*

Lemma 4.2. *If (M, ω) and $L \subset M$ are admissible then for any almost complex structure J compatible with ω , any J -holomorphic map from the disk to M that sends the boundary of the disk to L is constant.*

This follows from exactness in (ii) and (iv) and by the vanishing of relative π_2 in (iii). For (i) this is shown e.g. in [12, Thm. 1.2].

4.1. A family of quilts. In this section we introduce a family of quilts and in the next we use this family to define the maps on the hypercube. For more on quilts see [28].

Let $N > 0$ be a fixed integer and $\mathcal{E} = (\epsilon_1, \dots, \epsilon_N) \in \{-1, 1\}^N$. Put the lexicographic ordering on $\{0, 1\}^N$. Let $I, J \in \{0, 1\}^N$ be such that $I \leq J$. Let $R_{I,J}$ be the set of isomorphism classes of pairs (S_I, S_J) where S_I is the unit circle with a fixed point $z_+ := \sqrt{-1}$, called the *outgoing point*, and a finite set of marked points distinct from z_+ whose number is determined by I and J as follows. Give the circle minus z_+ the counterclockwise orientation. If $I_i < J_i$ and $\epsilon_i = +1$, a couple of adjacent points is associated to i and they are called a *pair*. This is subject to the following conditions.

- If $i < j$ then the points corresponding to i are before those corresponding to j in the counterclockwise orientation.
- If $I_i < J_i$ and $I_j < J_j$ with $j = i + 1$ then the left marked point corresponding to j and the right marked point corresponding to i are identified.

The marked circle S_J has a similar structure. The distinguished point on its boundary is called the *incoming point* and denoted by $z_- = -\sqrt{-1}$. If $\epsilon_i = -1$ and $I_i < J_i$, a couple of points on the circle minus z_- are assigned to i with the same identification and ordering requirement (now clockwise). $R_{I,J}$ is the quotient of the the set of such pairs of marked circles with the action of the Möbius transformations of the disc.

Since $R_{I,J}$ is a moduli space of points on two copies of the circle, it has the Deligne-Mumford-Stasheff compactification $\overline{R}_{I,J}$ by stable tree-discs as in Section 9f of [22]. Codimension one components of the boundary of $R_{I,J}$ correspond to two marked points (not necessarily adjacent) in S_I or S_J (but not both at once) converging together.

To $R_{I,J}$ corresponds a quilt $Q_{I,J}$ which can be described as follows. $Q_{I,J}$ can be thought of as lying in $[0, 2N + 1] \times \mathbb{R} \subset \mathbb{C}$ and satisfies the following; which give all the seams of Q . If $\epsilon_i = +1$

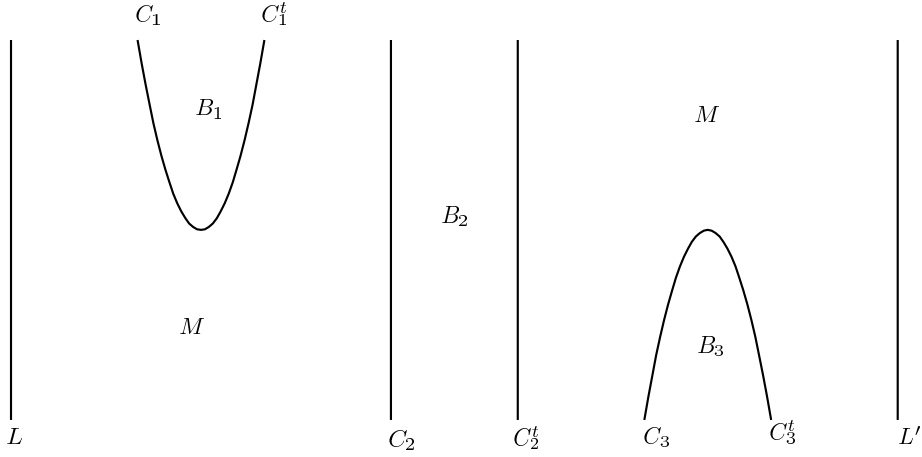


FIGURE 2. The quilt $Q_{I,J}$ for $\mathcal{E} = (-1, 1, 1)$ and $I = (0, 0, 0)$, $J = (1, 0, 1)$ with labelling.

then $Q \cap ([2i - 1, 2i] \times \mathbb{R})$ contains a semi-infinite cap if $I_i < J_i$ and is an infinite strip¹ if $I_i = J_i = 0$. If $\epsilon_i = -1$ then $Q \cap ([2i - 1, 2i] \times \mathbb{R})$ contains a semi-infinite cup if $I_i < J_i$ and is an infinite strip if $I_i = J_i = 1$. Figure 2 shows an example of such a quilt. For such a Q let Q^0 denote $[0, 2N + 1] \times \mathbb{R}$ minus all the cups, caps and strips mentioned above. Note that for any I , $Q_{I,I}$ consists of a single quilted strip as in Figure 3.

One assigns quilts to the boundary strata of $R_{I,J}$ as follows. Let r be a point in the codimension one stratum of $\partial R_{I,J}$. There is always a subinterval² $A \subset \{1, 2, \dots, N\}$ such that r corresponds to the converging of the marked points corresponding to A in S_I or S_J . Consider the case of S_I . Let $I' \in \{0, 1\}^N$ be obtained from I as follows. We set $I'_i = I_i$ if $i \notin A$. For $i \in A$, if $I_i = J_i$ we set $I'_i = I_i$ and if $I_i < J_i$ we set $I'_i = J_i$. It is evident that $I < I' < J$. To r we assign the pair of quilts $(Q_{I,I'}, Q_{I',J})$. You can see that gluing $Q_{I,I'}$ to the bottom of $Q_{I',J}$ gives back $Q_{I,J}$. The construction for the case that r corresponds to S_J is similar. The construction for the strata of codimension k is similar and gives a $(k + 1)$ -tuple of quilts (organized by a tree).

Since the quilts we consider have the same combinatorial type as polygons, the same method as that of Lemma 9.3 in [22] can be used to deduce the existence of a consistent choice of strip like ends for quilts associated to the strata of $\overline{R}_{I,J}$. This means that there exists a compact family $\mathcal{Q}_{I,J}$ of quilts and a projection

$$(12) \quad \pi : \mathcal{Q}_{I,J} \rightarrow \overline{R}_{I,J}$$

such that $\pi^{-1}(r)$ for any $r \in \overline{R}_{I,J}$ is a quilt with the combinatorial type described above. (So, for example for $r \in R_{I,J}$, $\pi^{-1}(r)$ has the same type as $Q_{I,J}$.) Concretely this involves choosing, for each $r \in \overline{R}_{I,J}$, the heights of the caps in cups in the quilt associated to r in such a way that when r converges to a point r' , the limiting heights agree with the ones given to the $\pi^{-1}(r')$ by gluing. One can also use Theorem 1.3 in [10] for this purpose.

¹i.e. $\{2i - 1\} \times \mathbb{R}$ and $\{2i\} \times \mathbb{R}$ are seams in Q .

²In the obvious sense.

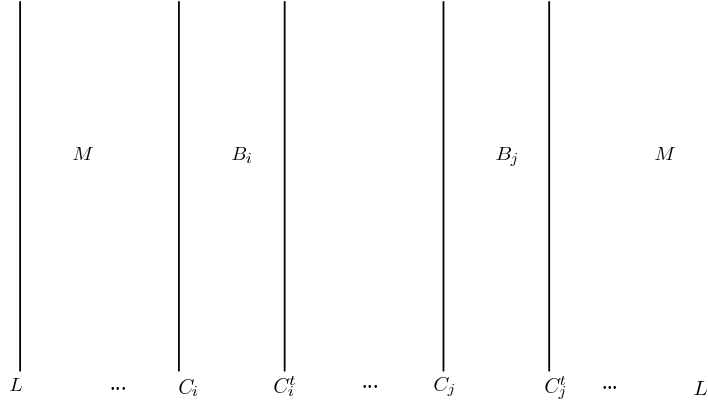


FIGURE 3. A quilted strip

Note that since the points of $\overline{R}_{I,J}$ are marked circles up to Möbius transformations, the points of $\mathcal{Q}_{I,J}$ are quilts up to vertical translation (i.e. Möbius transformations fixing positive and negative infinity). So for $N = 1$ the family of quilts is compact. Indeed for $I = (0)$ and $J = (1)$, $\mathcal{Q}_{I,J}$ consists of a single quilted triangle (pair of pants) as in Figure 1.

4.2. The maps of the hypercube. As before let M be a symplectic manifold, $L, L' \subset M$ Lagrangian submanifolds and C_1, \dots, C_N spheric coisotropic submanifolds of M such that C_i fibers over a symplectic manifold B_i . We label the seams of the quilts $\mathcal{Q}_{I,J}$ as follows. The boundary seams $\{0\} \times \mathbb{R}$ and $\{2N+1\} \times \mathbb{R}$ are labeled by L and L' respectively. For $i \in \{1, 2, \dots, N\}$ if $\{2i-1\} \times \mathbb{R}$ and $\{2i\} \times \mathbb{R}$ (or a subinterval of them) are part of seams in Q then they are labeled by C_{N-i+1} .

Recall [27] (also [16, section 3.1]) that a *generalized Lagrangian correspondence* (from point to itself) is a sequence

$$(13) \quad \mathbf{L} = (pt \xrightarrow{L_1} M_1 \xrightarrow{L_2} M_2 \rightarrow \dots \rightarrow M_k \xrightarrow{L_{k+1}} pt)$$

of symplectic manifolds (M_i, ω_i) and Lagrangian correspondences between them $L_i \subset M_{i-1}^- \times M_i$. By adding a diagonal correspondence $\Delta_{M_i} \subset M_i^- \times M_i$ if necessary we can assume that the number k is odd. Set $\mathcal{L}_0 = L_2 \times L_4 \times \dots \times L_{k+1}$ and $\mathcal{L}_1 = L_1 \times L_3 \times \dots \times L_k$ which are Lagrangian submanifolds of $\prod_i M_i$ (with the symplectic form $\sum (-1)^i \omega_i$). We say that the Lagrangian correspondence (13) *intersects transversely* if \mathcal{L}_0 and \mathcal{L}_1 intersect transversely. If this is the case and the L_i are compact then the intersection $\mathcal{L}_0 \cap \mathcal{L}_1$ is a finite set and its elements are the *generators* of the quilted Floer chain group $CF(\mathbf{L})$. These generators can be described as k -tuples $(x_1, \dots, x_k) \subset M_1 \times M_2 \times \dots \times M_k$ such that $(x_i, x_{i+1}) \in L_{i+1}$.

Let $I, J \in \{0, 1\}^N$ be such that $I \leq J$. In order to define the maps between the vertices (2) of the hypercube, corresponding to I and J we first need to choose Hamiltonian isotopies that make each one of the generalized correspondences $(L, C_N^{I_N}, C_{N-1}^{I_{N-1}}, \dots, C_1^{I_1}, L')$ and $(L, C_N^{J_N}, C_{N-1}^{J_{N-1}}, \dots, C_1^{J_1}, L')$ intersect transversely. We also need to choose for each $\mathcal{Q}_{I,J} \in \mathcal{Q}_{I,J}$ a family of compatible almost complex structures on M and the B_i , parameterized by the points³ of each patch of $\mathcal{Q}_{I,J}$

³More precisely parameterized by the x coordinate of the points outside a compact subset.

which make the linearization of the equation of pseudoholomorphic maps surjective (and so yield smooth moduli spaces). In addition such choices for $Q_{I,J}$, $Q_{J,K}$ and $Q_{I,K}$ with $I \leq J \leq K$ must be consistent under gluing. This is done in an inductive way from highest codimension strata of $Q_{I,J}$ (which consist of quilted strips and/or quilted pairs of pants) up to its interior. See Theorem 6.24 in [10]. Such a set of Hamiltonian isotopies and almost complex structures is called a *regular set of perturbation data*. In the sequel we assume that such Hamiltonian isotopies and almost complex structures are chosen and are used to define the set of generators and the equation for pseudoholomorphic curves.

For each I, J with $I \leq J$ denote by $\mathcal{M}_{I,J}$ the moduli space of pairs (Q, \underline{u}) where $Q \in \mathcal{Q}_{I,J}$ and $\underline{u} = (u_0, \dots, u_N)$ is such that

- for $j > 0$, if $Q \cap [2j-1, 2j] \times \mathbb{R}$ is a cap, a cup or a strip, u_j is a pseudoholomorphic map from that cap/cup/strip to B_j (the base of the fibration of the coisotropic C_j) and is a space holder otherwise.
- $u_0 : Q^0 \rightarrow M$ is pseudoholomorphic.
- Whenever there is a seam between two components of Q and u_j, u_k are their corresponding pseudoholomorphic maps then the pair (u_j, u_k) sends the seam to the Lagrangian correspondence labeling the seam.

Outside a compact set each quilt is a quilted strip (as in Figure 3) which is the same thing as a strip in the product manifold with boundary on the product Lagrangians L_0, L_1 . Therefore pseudoholomorphic quilts have the exponential convergence property which means that each such \underline{u} maps $-\infty$ to a generator of CF_I and $+\infty$ to a generator of CF_J . So, $\mathcal{M}_{I,J}$ is a disjoint union of $\mathcal{M}_{I,J}(\underline{x}, \underline{y})$ for $\underline{x}, \underline{y}$ generators of CF_I and CF_J respectively. $\mathcal{M}_{I,J}(x, y)$ gets a topology from the topology of $R_{I,J}$ and the fact that for $i > 0$ (resp. $i = 0$), u_i lies in the $W^{1,2}$ space of maps from the strip/cap/cup (resp. Q^0) to B_i (resp. M). The regular choice of almost complex structures on M and the B_i implies that $\mathcal{M}_{I,J}(x, y)$ is a disjoint union of smooth manifolds⁴ of possibly different dimensions. Let $\mathcal{M}_{I,J}(\underline{x}, \underline{y})_d$ denote the d dimensional part of $\mathcal{M}_{I,J}(\underline{x}, \underline{y})$. If $\underline{u} = (u_1, \dots, u_N)$ then by definition the energy of \underline{u} is the sum of the energies of the u_i .

Lemma 4.3. *For each d there is a constant k_d such that the energy of each \underline{u} , for (Q, \underline{u}) element of $\mathcal{M}_{I,J}(x, y)_d$, is less than k_d .*

Proof. For exact Lagrangians this follows easily from Stokes theorem applied to the pullback of the appropriate symplectic forms by the u_i . For other types we note that as in the proof of Theorem 3.9 in [28], near each interior seam the couple of maps (u_i, u_j) (where either $j = 0$ or $j = i + 1$) is equivalent to a map u'_i from the strip into the product $M^- \times B_i$ which sends one boundary component to C_i . This is the case because the seams are assumed to be real analytic (and we are negating the symplectic form on B_i). Note that for the quilts that we use this neighborhoods are dense in each component of the quilt. This way, two $\underline{u}, \underline{v}$ give, for each i , a map from the cylinder $S^1 \times \mathbb{R}$ to $M^- \times B_i$ whose area is the difference of the energies of u'_i and v'_i . Therefore the lemma will follow if we show that the energy of any pseudoholomorphic map $w : S^1 \times [0, 1] \rightarrow M^- \times B_i$, which sends $S^1 \times \{1\}$ to C_i , is proportional to the Maslov index of the curve $w|_{S^1 \times \{1\}}$ i.e. $E(w) = k \cdot \mu(w|_{S^1 \times \{1\}})$ where k is the same for all the C_i and L, L' . This is because this Maslov index has

⁴More precisely this means that $\mathcal{M}_{I,J}(x, y)$ is locally a smooth submanifold of the Cartesian product of $R_{I,J}$ with the space of $W^{1,2}$ maps.

to be less than a number which does not depend on the curves otherwise \underline{u} and \underline{v} would not be in the d dimensional part of the moduli space.

If L, L' and the C_i are admissible of type (i), the assumption on the fundamental groups implies that each such w is homotopic to a map of the disk and then one can use the monotonicity of C_i, L or L' . If the Lagrangian and coisotropic submanifolds admissible of type (iii), this is shown in [27, Lemma 4.1.5] \square

Lemma 4.4. *If M, L, L' and the B_i satisfy the Admissibility Condition (iv) then the elements of $\mathcal{M}_{I,J}(x, y)$ lie in a compact subset of M .*

Proof. It is shown in [18, Lemma 3.3.2], using an argument of Oh [13], that holomorphic curves with boundary on two exact Lagrangian submanifolds of a Stein manifold satisfying admissibility condition of type (iv), lie in a compact set determined (only) by the intersection points of the Lagrangians. One generalizes this to pseudoholomorphic quilts using the same “folding” argument used in Lemma 4.3. \square

Lemma 4.5. *If M, L, L' and the B_i satisfy the Admissibility Condition then $\mathcal{M}_{I,J}(x, y)_0$ is a finite set.*

Proof. Let (Q_n, \underline{u}_n) be a sequence of elements of $\mathcal{M}_{I,J}(x, y)_0$. If Q_n converges to an element of the boundary of $\mathcal{Q}_{I,J}$, which for simplicity we assume to be in the codimension one part and so of the form $(Q_{I,I'}, Q_{I',J})$ for $I < I' < J$, then because of the boundedness of energy Lemma 4.3, u_n will converge, up to reparametrization, to a pair $(u_{I,I'}, u_{I',J})$ where $u_{I,I'} \in \mathcal{M}_{I,I'}$ and $u_{I',J} \in \mathcal{M}_{I',J}$. So by the gluing theorem for pseudoholomorphic quilts (Proved in [11, Theorem 1] for pseudoholomorphic quilted polygons) there is a one parameter family of pseudoholomorphic quilts and this, together with the fact that $\mathcal{M}_{I,J}(x, y)$ is locally a manifold, contradicts the assumption that (Q_n, u_n) lie being in the zero dimensional part of $\mathcal{M}_{I,J}(x, y)$. The same argument prevents the limit from being a “broken quilt” i.e. of the form $(u_{I,I}, u_{I,J})$ or $(u_{I,J}, u_{J,J})$. The only other possible limit is a bubbling of a boundary disk which is ruled out by Lemma 4.2. Therefore $\mathcal{M}_{I,J}(x, y)_0$ is compact. \square

Therefore one can define a linear map $\mu_{I,J} : CF_I \rightarrow CF_J$ given on the generators of CF_I by

$$(14) \quad \mu_{I,J}(\underline{x}) = \sum_{\underline{y}} \# \mathcal{M}_{I,J}(\underline{x}, \underline{y})_0 \cdot \underline{y}$$

where the sum is over the generators of CF_J . Note that $\mu_{I,I}$ is the (quilted) Floer differential on CF_I . Also if the manifolds B_i are points and the ϵ_i are positive then the maps $\mu_{I,J}$ are higher composition maps in the Fukaya category $\mathcal{F}(M)$ of M .

Define a linear map $D : CF_{\oplus} \rightarrow CF_{\oplus}$ by

$$(15) \quad D = \sum_{I \leq J} \mu_{I,J}.$$

Let G_I denote the set of generators of CF_I .

Lemma 4.6. *If M, L, L' and the C_i satisfy the Admissibility Condition then $D^2 = 0$.*

Proof. This is a special case of the Master Equation for quilt families [10, Theorem 1.5]. We note that if $\underline{x} \in G_I$ then

$$(16) \quad D^2 \underline{x} = \sum_{J \geq I} \sum_{K \geq J} \sum_{\underline{y} \in G_J} \sum_{\underline{z} \in G_K} \# \mathcal{M}_{I,J}(\underline{x}, \underline{y})_0 \times \mathcal{M}_{J,K}(\underline{y}, \underline{z})_0 \cdot \underline{z}.$$

Therefore if we show that $\partial\mathcal{M}_{I,K}(\underline{x}, \underline{z})_1$ can be identified with

$$(17) \quad \bigcup_{I \leq J \leq K} \bigcup_{\underline{y} \in G_J} \mathcal{M}_{I,J}(\underline{x}, \underline{y})_0 \times M_{J,K}(\underline{y}, \underline{z})_0$$

the result follows. The proof is standard and similar to that of Lemma 4.5. If (Q_n, u_n) is a sequence in $\mathcal{M}_{I,K}(\underline{x}, \underline{y})_1$ then, by the gluing argument, the limit of $\{Q_n\}$ can not lie in the part of $\partial R_{I,J}$ of codimension two or higher because then the u_n would not be in the one dimensional part of $\mathcal{M}_{I,K}(\underline{x}, \underline{y})$. So, the limit of Q_n , if not in the interior of $R_{I,J}$, is a pair $(Q_{I,J}, Q_{J,K})$ with $I \leq J \leq K$. Therefore u_n converges, up to reparametrization, to a pair $(u_{I,J}, u_{J,K})$ where $u_{I,J} \in \mathcal{M}_{I,J}$ and $u_{J,K} \in \mathcal{M}_{J,K}$ have to be in the zero dimensional part. It follows from exponential convergence that $u_{I,J}$ (resp. $u_{J,K}$) sends $+\infty$ (resp. $-\infty$) to an element of G_J which have to be the same. Bubbling is again ruled out by 4.2. Therefore $\partial\mathcal{M}_{I,K}(\underline{x}, \underline{y})_1$ is included in (17). The reverse inclusion follows from the gluing argument for pseudoholomorphic quilts [11]. \square

5. THE WEHRHEIM-WOODWARD EXACT TRIANGLE

The main tool that we use in this paper to prove Theorem 6.1 is the following result of Wehrheim and Woodward. See also [16] and [4] for related results. Let M be a symplectic manifold and $L, L' \subset M$ be Lagrangian.

Theorem 5.1 (Wehrheim-Woodward[25], Theorem 5.2.9). *Let $C \subset M$ be a spheric coisotropic submanifold fibering over a manifold B . If the triple (C, L, L') is monotone and each one of C, L, L' has Maslov index at least 3 then $CF(L, \tau_C, L')$ is quasi-isomorphic to the cone of*

$$(18) \quad CF(L, C, C^t, L')[\dim_C B] \xrightarrow{\mu} CF(L, L')$$

where μ is given by the count of pseudoholomorphic quilted triangles as in Figure 1 i.e. $\mu = \mu_{(0),(1)}$.

The theorem more generally holds for L, L' generalized Lagrangian submanifolds of M (i.e. generalized correspondences between M and a point). See [18, Prop. 5.16]. We now recall the construction of the quasi-isomorphism. For simplicity denote $C_0 = CF(L, C, C^t, L')$, $C_1 = CF(L, L')$ and $C_\infty = CF(L, \tau_C, L')$. It is easy to see that, in general, a chain map from $\text{Cone}(f)$ to C_∞ consists of a chain map $k : C_1 \rightarrow C_\infty$ and a homotopy $h : C_0 \rightarrow C_\infty$ between $k \circ \mu$ and zero. Moreover (h, k) is a quasi-isomorphism if and only if its mapping cone is acyclic. The maps h, k are given by counting pseudoholomorphic sections of “quilted Lefschetz-Bott fibrations” in Figures 4 and 5. The map k is given by the count of the rigid sections of the quilted fibration of Figure 4.

Let Q_t for $t \in [0, 1]$ be the one parameter family of quilted fibrations such that Q_0 is the fibration in Figure 5 and as $t \rightarrow 1$, the critical value approaches the cap and a circle is pinched off the cap. The map h is given by the count of zero dimensional part of the moduli space $\{(Q_t, s) | t \in [0, 1]\}$ where s is a section of the fibration Q_t . By definition (and because bubbling is ruled out) the map h is a homotopy between $k \circ \mu$ and the map associated to Q_1 which we denote by h_1 . Wehrheim and Woodward [25, Lemma 5.2.1] show that if the codimension of C is ≥ 2 then h_1 is zero and hence (h, k) is a chain map from $\text{Cone}(\mu)$ to $CF(L, \tau_C, L')$.

In fact, with coefficients in $\mathbb{Z}/2$, this holds for the codimension one case as well. To see this we recall the proof of the vanishing of h_1 . Let E_C denote the Lefschetz-Bott fibration over the disk whose vanishing cycle is C . (Such a fibration always exists; see [15, 2.4.1].) The quilted fibration Q_1 that gives h_1 is the result of gluing E_C to another quilted Lefschetz-Bott fibration. It follows from gluing theorem (along a seam) for pseudoholomorphic quilts that if the moduli space of pseudoholomorphic sections of E_C has sufficiently high dimension near any point then the zero

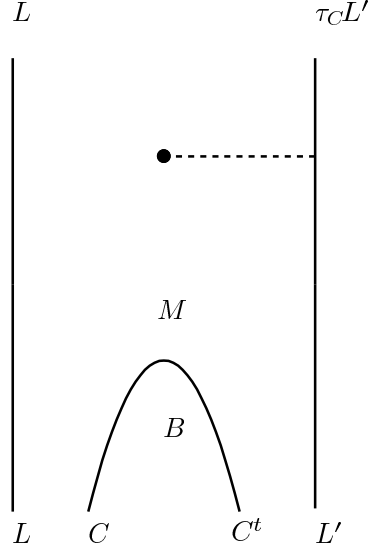
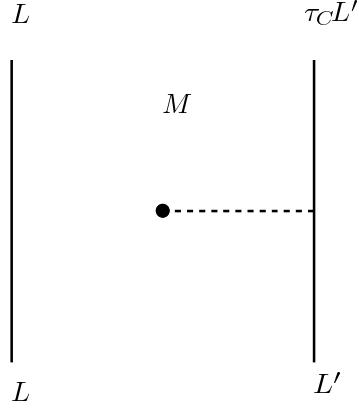


FIGURE 4. The quilted fibration used to construct the mas $C_1 \rightarrow C_\infty$. One considers a (unique up to isotopy) Lefschetz-Bott fibration over each patch of the quilt whose fiber over any point is either M or B as indicated. The dot represents the unique critical value of the fibration. The fibration is such that its monodromy around the critical point is Hamiltonian isotopic to τ_C . A *section* of such quilted fibration consists of a pair of pseudoholomorphic sections (of the fibrations over each patch) whose value on each seam lie in the given Lagrangian submanifold.

dimensional part of the moduli space of sections of Q_1 is empty. This is indeed what Wehrheim and Woodward do i.e. (as in [21]) they explicitly produce a high dimensional moduli of sections of E_C containing a given section. For the codimension one case, as in [22, 17.15] (for Lefschetz fibrations), the zero dimensional part of the moduli space of sections of such a fibration consists of two points and hence h_0 is zero over $\mathbb{Z}/2$. The same result should hold over \mathbb{Z} with a careful choice of signs as in the aforementioned reference however we do not use this.

To prove the acyclicity of $\text{Cone}(h, k)$ as in [21] Wehrheim and Woodward use Floer homology over the Novikov ring (even though the Lagrangians are monotone). Such chain complexes are filtered by area. Let D denote the differential on $\text{Cone}(h, k)$. One of the main ingredients in the proof is the decomposition of D into two parts $D = D_0 + D_1$ whose degrees have a gap in between i.e. the degree of D_0 is in $[0, \epsilon)$ and the degree of D_1 is in $[2\epsilon, \infty)$ for some $\epsilon > 0$. The other ingredient is using a geometric argument to prove the acyclicity of D_0 for fibered Dehn twists with small support. The acyclicity of $\text{Cone}(h, k)$ follows from these two by homological algebraic arguments.

FIGURE 5. The quilted fibration used to construct the map $C_0 \rightarrow C_\infty$.

As a consequence one has an exact triangle

$$\begin{array}{ccc}
 & HF(L, \tau_C L') & \\
 [1] \swarrow & & \nwarrow k \\
 HF(L, C^t, C, L')[\dim_{\mathbb{C}} B] & \xrightarrow{\mu} & HF(L, L')
 \end{array}$$

where the map $\xrightarrow{[1]}$ is the connecting homomorphism.

Remark 5.2. One has a corresponding exact triangle for negative Dehn twists. We have $CF^i(L, \tau_C^{-1} L') \simeq CF^i(\tau_C L, L') \simeq \text{Hom}(CF^{l-i}(L', \tau_C L), \mathbb{Z}/2)$ where $l = \dim L$. This induces an isomorphism on Floer homology. Noting that the map induced by the dual Q^t of a quilt Q is the dual of the map induced by Q , Theorem 5.1 yields a quasi-isomorphism from $CF(L, \tau_C^{-1} L')$ to $\text{Cone}(\mu^t)[-1]$ where $\mu^t : CF(L, L') \rightarrow CF(L, C^t, C, L')$ is given by counting the upside-down version of the quilt in Figure 1. This results in an exact triangle as follows.

$$\begin{array}{ccc}
 & HF(L, \tau_C^{-1} L') & \\
 [1] \swarrow & & \nwarrow k^t \\
 HF(L, C^t, C, L')[\dim_{\mathbb{C}} B - 1] & \xleftarrow{\mu^t} & HF(L, L')[-1]
 \end{array}$$

Corollary 5.3. If M and L, L', C are admissible then the conclusion of Theorem 5.1 holds.

Proof. The assumption of Maslov index at least 3 in Theorem 5.1 is to rule out disc bubbling. In our case bubbling is ruled out by Lemma 4.2. Monotonicity of (C, L, L') for (i) follows from [27, Lemma 4.1.3], for (iii) from [27, Lemma 4.1.5] and for (ii) and (iv) from exactness. Finally for (iv) we need to show that all the pseudoholomorphic curves involved lie in a compact subset of M . This was shown in [18, proposition 5.14]. (See Lemma 4.4.) \square

Here we make a simple observation about the map μ used in the exact triangle which is useful in computations. Let $\pi : C \rightarrow B$ be the projection. Also let α, β, γ be the components of the boundary of a disk with three punctures.

Proposition 5.4. *For a regular family of almost complex structures, the map μ is given by (the zero dimensional part of) the moduli space of pseudoholomorphic maps from the disk into M which send α, β, γ respectively to L, L', C and for which $\pi \circ u|_\gamma$ can be completed to a pseudoholomorphic disk in B .*

Proof. Let u and v be pseudoholomorphic maps from the two components of Q into M and B respectively. By the abuse of notation let γ denote the seam between the two components of this quilt. Since C is embedded in M , we have $v|_\gamma = \pi \circ u|_\gamma$ where $\pi : C \rightarrow B$ is the projection. By unique continuation for pseudoholomorphic maps [1], v is uniquely determined by $v|_\gamma$ and therefore by u . \square

6. THE SPECTRAL SEQUENCE

In this section we show that the chain complex of (1) is quasi-isomorphic to the hypercube CF_\oplus of resolutions of the twists constructed in the last section. We do this by an inductive argument involving the exact triangle of Wehrheim and Woodward. We use the abbreviation

$$(19) \quad CF := CF(L, \tau_{C_N}^{\epsilon_N} \circ \tau_{C_{N-1}}^{\epsilon_{N-1}} \circ \cdots \circ \tau_{C_1}^{\epsilon_1}(L')).$$

Theorem 1.1 follows from the following.

Theorem 6.1. *Assume L, L' and the C_i satisfy the Admissibility Condition. Then CF_\oplus and CF are isomorphic in the derived category $D^b(\mathbb{Z}/2\text{-mod})$. In other words there is a chain complex CF^+ and chain maps $f : CF^+ \rightarrow CF_\oplus$, $g : CF^+ \rightarrow CF$ which induce isomorphisms on homology. Moreover if the Floer homology groups are graded, this quasi-isomorphism preserves the grading.*

Before proceeding to the proof, we recall the generalities about grading on Lagrangian Floer homology [20]. Let (M, ω, J) be a symplectic manifold with a compatible almost complex structure and assume there is an $n > 0$ such that $2c_1(M)$ is zero in $H^2(M, \mathbb{Z}/n)$. This implies that there is a (non-unique) line bundle η and an isomorphism $r : \eta^{\otimes n} \rightarrow \Lambda^{\max}(TM, J)^{\otimes 2}$. Let $\text{Lag}(M)$ denote the fiber bundle over M whose fiber over any point x is the Lagrangian Grassmannian of $T_x M$. The isomorphism classes of such pairs (η, r) is in one to one correspondence with fiber bundles $\mathbb{L} \rightarrow M$ whose fiber over each point x is an n fold cover of the Lagrangian Grassmannian of $T_x M$ (corresponding to the Maslov class) together with a map $\mathbb{L} \rightarrow \text{Lag}(M)$ which is an n fold covering. For each Lagrangian submanifold $L \subset M$ there is a canonical section s_L of $\text{Lag}(M)$. A grading on L is a lift of this section to a section of \mathbb{L} . A choice of grading for two Lagrangians L, L' induces an absolute \mathbb{Z}/n grading on $HF(L, L')$.

The quasi-isomorphisms f, g are given explicitly in terms of maps induced by quilted fibrations. As the first step we note that CF is quasi-isomorphic to

$$(20) \quad CF(L, \text{graph } \tau_{C_N}^{\epsilon_N}, \dots, \text{graph } \tau_{C_1}^{\epsilon_1}, L').$$

The chain complex CF^+ is the hypercube associated to resolving only the positive twists, i.e. ones for which $\epsilon_i = +1$. Let n_+ denote the number of positive twists in (20) and for $I \in \{0, 1\}^{n_+}$ let C_I^+ be the Floer chain complex for the correspondence obtained from $(L, \text{graph } \tau_{C_N}^{\epsilon_N}, \dots, \text{graph } \tau_{C_1}^{\epsilon_1}, L')$

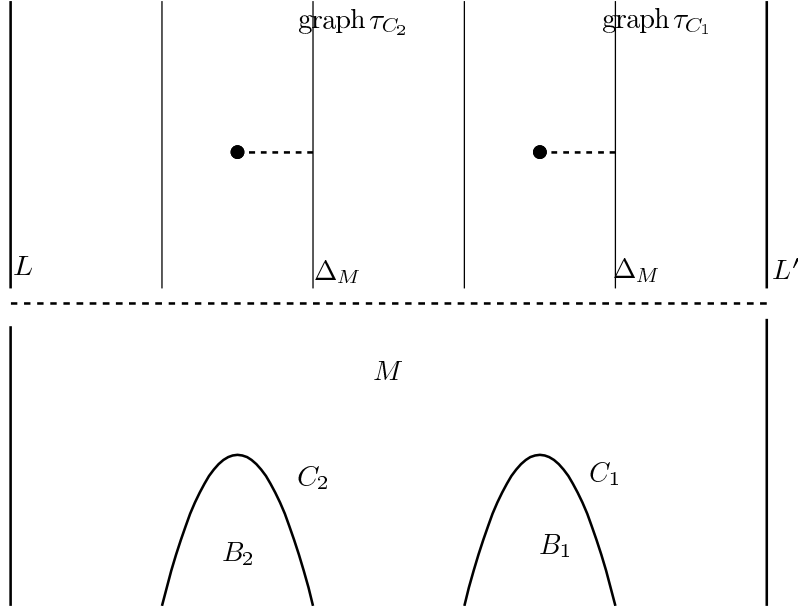


FIGURE 6. The family of quilted fibrations $\mathcal{Q}_{1,\infty}$ (top) which is used in conjunction with the family $\mathcal{Q}_{I,1}$ from section 4.1 (bottom) to define the map f . The y coordinates of the critical values of the fibration (which are denoted by dots) can vary but they are bounded above.

in which the i th positive fibered twist is replaced with its I_i -resolution. Set

$$(21) \quad CF^+ = \bigoplus_{I \in \{0,1\}^{n_+}} CF_I^+.$$

One can define maps $\mu_{I,J} : CF_I^+ \rightarrow CF_J^+$ for $I \leq J$ as before and the sum of all such $\mu_{I,J}$ makes CF^+ into a chain complex.

Now we describe the map f . The map g has a dual description (as in Remark 5.2). Let $\mathbf{1} = (1, 1, \dots, 1) \in \{0, 1\}^{n_+}$. Consider the family of quilted fibrations as in Figure 6 in which the y coordinates of the critical values are arbitrary but bounded above. Denote this family by $\mathcal{Q}_{1,\infty}$. The quilt family that gives the map $f_I : CF_I^+ \rightarrow CF$ is obtained by putting the quilts in the family $\mathcal{Q}_{1,\infty}$ on top of the quilts in the family $\mathcal{Q}_{I,1}$ from section 4.1 (but now with n_+ in place of N). To show that $f = \sum_{I \in \{0,1\}^{n_+}} f_I$ is a chain map, since bubbling is ruled out by the Admissibility Condition, one inspects the limits of the family of quilts used. As in the case $n_+ = 1$ (section 5), when the y coordinate of a critical value goes to $-\infty$, the resulting map is zero. The remaining boundary components give the equation $\sum_{J \leq I} f_J \circ \mu_{I,J} = D \circ f_I$ where D is the differential on CF . Note that for $n_+ = 1$, f is the map (h, k) from section 5.

Proof. (of Theorem 6.1) We only prove that f is a quasi-isomorphism by induction on n_+ . The proof for g is similar. By Theorem 5.1 there is a chain map (h, k) from the cone of $\mu_{(0),(1)} : CF_0 \rightarrow CF_1$ to CF where

$$CF_0 = CF(L, C_N^t C_N, \tau_{C_{N-1}} \circ \dots \circ \tau_{C_1}(L))$$

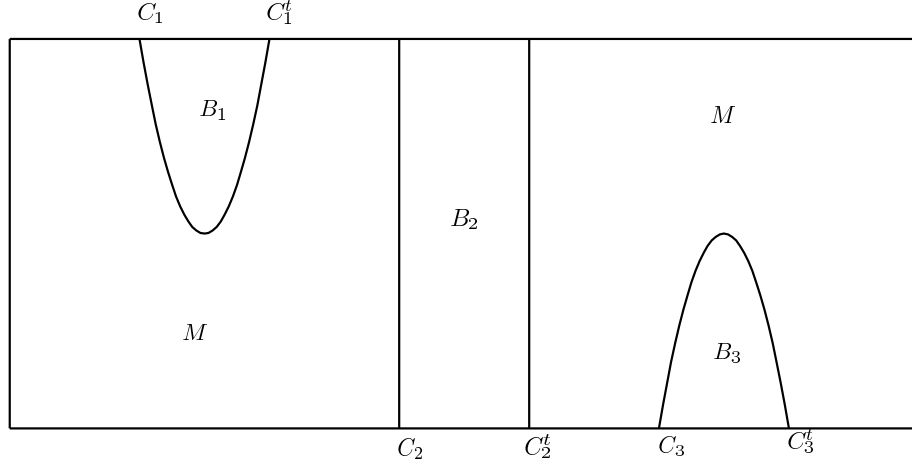


FIGURE 7. An example of the quilts used in constructing the hypercube in $\mathcal{F}(M, M)$. The boundary rectangle represents a cylindrical end.

and $CF_1 = CF(L, \Delta_M, \tau_{C_{N-1}} \circ \cdots \circ \tau_{C_1}(L))$ and (h, k) induces an isomorphism on cohomology. We use the shorthand notation $iI = (i, I_1, \dots, I_{N-1})$ for $i = 0, 1$ and set $CF_{i\oplus} = \bigoplus_{I \in \{0,1\}^{N-1}} CF_{iI}$. By the induction hypothesis we have quasi-isomorphisms $f_0 : CF_{0\oplus} \rightarrow CF_0$ and $f_1 : CF_{1\oplus} \rightarrow CF_1$. We construct a chain map $\nu : CF_{0\oplus} \rightarrow CF_{1\oplus}$ which makes the following diagram commutative up to homotopy.

$$(22) \quad \begin{array}{ccc} CF_0 & \xrightarrow{\mu_{(0),(1)}} & CF_1 \\ f_0 \uparrow & & \uparrow f_0 \\ CF_{0\oplus} & \xrightarrow{\nu} & CF_{1\oplus} \end{array}$$

Set $\nu_{I,J} = \mu_{0I,1J}$. That ν is a chain map is a consequence of Lemma 4.6. More precisely if $D_{i\oplus}$ denotes the differential on $CF_{i\oplus}$ then $\nu D_{0\oplus} - D_{1\oplus} \nu = D^2 - D_{0\oplus}^2 + D_{1\oplus}^2$. Because the maps f_0, f_1 are given by counting quilts, an argument similar to that of Lemma 4.6 shows that each ν makes the diagram (22) commutative up to homotopy. This is again a special case of the master equation for family quilt invariants. Therefore (f_0, f_1) gives a quasi-isomorphism from $\text{Cone}(\nu) = CF_{\oplus}$ to $\text{Cone}(\mu_{(0),(1)})$. Now it is easy to see that the composition $(h, k) \circ (f_0, f_1)$ is given by counting the sections of the a family of quilted fibrations isotopic to the one we used to define f . Therefore they induce the same map on homology. \square

Remark 6.2. The question of whether the spectral sequence of Theorem 1.1 collapses at the E_2 level is related to the formality of the (generalized) Fukaya category of M . Since the E_1 page is always doubly graded, formality would follow from the existence of a second grading on the Fukaya category which is preserved by the higher composition maps. One can expect that such extra grading should come from extra geometric structures on the manifold e.g. a hyperkähler structure or a circle action.

Note however that the spectral sequence will always collapse at the E_{N+1} term because of the finiteness of the filtration.

For $I \in \{0, 1\}^N$ denote $\mathbf{C}^I = (C_N^{I_N}, \dots, C_1^{I_1})$ which is a Lagrangian correspondence from M to itself. Assuming the Admissibility Condition is satisfied by the C_i , let $\tilde{\mu}_{I,J} \in CF(\mathbf{C}^I, \mathbf{C}^J)$ be given by the count of quilted disks with cylindrical ends which are obtained from quilt $Q_{I,J}$ as in Figure 7.

Proposition 6.3. *Under the same assumptions as in Theorem 6.1, the Lagrangian correspondence*

$$(23) \quad \text{graph}(\tau_{C_N}^{\epsilon_N} \circ \dots \circ \tau_{C_1}^{\epsilon_1})$$

is isomorphic, in $D\mathcal{F}^\#(M, M)$, to $(\sum_I \mathbf{C}^I, \sum_{I \leq J} \tilde{\mu}_{I,J})$.

Proof is the same as that of Theorem 6.1 but with quilts of the form in Figure 7.

6.1. Naturality of the spectral sequence. Since the quasi-isomorphism ϕ of Theorem 6.1 is given by quilt maps, it satisfies a form of naturality under equivalence of Lagrangian correspondences as follows. First recall that if $L_0 \subset M_0 \times M_1$ and $L_1 \subset M_1 \times M_2$ are two Lagrangian correspondences then their composition $L_1 \circ L_0 \subset M_0 \times M_2$ is

$$\{(m_0, m_2) \mid \exists m_1 \in M_1 \text{ s.t. } (m_0, m_1) \in L_0 \text{ \& } (m_1, m_2) \in L_1\}.$$

It can be equally described as the intersection of $L_0 \times L_1$ with $M_0 \times \Delta_{M_1} \times M_2$ in $M_0 \times M_1 \times M_1 \times M_2$. We say that the composition of L_0 and L_1 is *embedded* if this intersection is transversal and its projection into $M_0 \times M_1$ is an embedding.

Let

$$(24) \quad \mathbf{L} = (L_1, L_2, \dots, L_n)$$

be a generalized Lagrangian correspondence. Assume further that for some $1 \leq k \leq n-1$, the composition $L_{k+1} \circ L_k$ is embedded. Then \mathbf{L} and

$$(25) \quad \mathbf{L}' = (L_1, \dots, L_{k-1}, L_{k+1} \circ L_k, L_{k+2}, \dots, L_n)$$

are said to be equivalent. More generally \mathbf{L} and \mathbf{L}' are said to be *equivalent in the symplectic category* if they are related by a sequence of such moves. It is not difficult to see that the generators of $CF(\mathbf{L})$ and $CF(\mathbf{L}')$ are in one to one correspondence. The functoriality theorem of Wehrheim and Woodward [26, Thm. 1.0.1] (together with the discussions of monotonicity in [27]) implies that if \mathbf{L} and \mathbf{L}' satisfy the Admissibility Condition then their Floer homologies are canonically isomorphic. Moreover such isomorphisms are compatible with maps induced by quilts. More precisely we have the following result (which is stated in greater generality in [28]).

Theorem 6.4 (Wehrheim, Woodward [28], Thm. 5.1). *Let Q be a quilt with one incoming and one outgoing end. Assume that the incoming (resp. outgoing) end is labeled by \mathbf{L}_0 (resp. \mathbf{L}_1). Assume that Q has a strip in it whose seams are labeled by correspondences L_k and L_{k+1} whose composition is embedded. Let Q' be the quilt obtained from Q by removing the this strip and replacing it with a seam labeled by $L_{k+1} \circ L_k$. Let $\mathbf{L}'_0, \mathbf{L}'_1$ be the labellings of the incoming and outgoing ends of Q' respectively. Then one has a commutative diagram*

$$(26) \quad \begin{array}{ccc} CF(\mathbf{L}_1) & \longrightarrow & CF(\mathbf{L}'_1) \\ \uparrow \Phi(Q) & & \uparrow \Phi(Q') \\ CF(\mathbf{L}_0) & \longrightarrow & CF(\mathbf{L}'_0) \end{array}$$

where the horizontal maps are isomorphisms and vertical ones are quilt maps. Furthermore if $\mathbf{L}_0, \mathbf{L}_1$ satisfy the Admissibility Condition (i) the horizontal maps induce isomorphisms on homology.

Note that Lemma 4.2 together with the discussions of monotonicity in [28] imply that the above theorem holds if L_i satisfy the Admissibility Condition.

Proposition 6.5. *Let L_i, L'_i be two generalized submanifolds of M which are equivalent in the symplectic category for $i = 0, 1$, and D, D' two equivalent generalized correspondence from M to itself. Finally let ψ_0, ψ_1 be two symplectomorphisms of M given by compositions of fibered Dehn twists along spheric coisotropic submanifolds of M . If all these satisfy the Admissibility Condition then we have a commutative diagram*

$$(27) \quad \begin{array}{ccc} CF(L_0, \psi_0, D, \psi_1, L_1) & \longrightarrow & CF(L'_0, \psi_0, D', \psi_1, L'_1) \\ \uparrow & & \uparrow \\ CF_{\oplus}(L_0, \psi_0, D, \psi_1, L_1) & \longrightarrow & CF_{\oplus}(L'_0, \psi_0, D', \psi_1, L'_1) \end{array}$$

where the horizontal maps are natural isomorphisms which induce isomorphisms on homology. The vertical ones are isomorphisms (in the derived category) given by Theorem 6.1.

The proof is an application of Theorem 6.4.

7. APPLICATIONS

7.1. Symplectic Khovanov homology. Symplectic Khovanov homology is an invariant of links introduced by Seidel and Smith [23]. It is expected to be equivalent to Khovanov homology. We use the same notation as in section 3.1. Let $z_j = (j, 0)$ for $j = 0, \dots, 2m$ be points in the plane. They give rise to a point $v \in Conf_{2m}$. Let $\delta_i : [0, 1) \rightarrow Conf_{2m}$ be a curve such that $\delta(0) = v$ and as $t \rightarrow 1$, z_i and z_{i+1} merge (but the other points remain fixed). As mentioned in section 3.1, this gives us locally defined spheric coisotropic submanifolds $L_i = L_{\delta_i} \subset \mathcal{Y}_{m,v}$ which fiber over a compact subset $U \subset \mathcal{Y}_{m-1,\mu}$ where $\mu = v \setminus \{z_i, z_{i+1}\}$. By composing the correspondences L_{2i-1} for $i = 1, \dots, m$ we get a Lagrangian submanifold $\mathcal{L} \subset \mathcal{Y}_{m,v}$ which does not depend on the choice of the open sets U (because it is compact). We set the compact subset $U \subset \mathcal{Y}_{m-1,\mu}$ to contain the projection of a neighborhood of \mathcal{L} . Even though fibered Dehn twists along the L_j are defined locally, since \mathcal{L} is compact and the Dehn twists are compatible (up to Hamiltonian isotopy) under enlargement of U , the image of \mathcal{L} under a composition of such Dehn twists is a well-defined Lagrangian submanifold up to Hamiltonian isotopy.

Following the (slight) reformulation in [19] let a link K be given as the plat closure of a braid $\beta \in Br_{2m}$ which in turn is given by a braid word $\sigma_{k_1}^{\epsilon_1} \sigma_{k_2}^{\epsilon_2} \cdots \sigma_{k_N}^{\epsilon_N}$. Let w denote the writhe of this braid. One has

$$(28) \quad Kh_s(K) = HF(\mathcal{L}, \tau_{L_{k_1}}^{\epsilon_1} \circ \cdots \circ \tau_{L_{k_N}}^{\epsilon_N}(\mathcal{L}))[-m-w].$$

Remark 7.1. *Seidel and Smith define their invariant using “rescaled” monodromy maps of the fibration χ . The equivalence with the above formulation follows from a theorem of Perutz [15, Theorem 2.19], which states that the monodromy of a normally Kähler Lefschetz-Bott fibration around a critical value is Hamiltonian isotopic to fibered Dehn twist along the vanishing cycle of the fibration, and the fact that the Lagrangian \mathcal{L} is compact.*

The Lagrangian correspondences L_i and the Lagrangian submanifold \mathcal{L} are exact because the symplectic form on $\mathcal{Y}_{m,v}$ is exact and the fibers of the fibration $L_i \rightarrow \mathcal{Y}_{m-1,v}$ are simply connected. It was shown in [19, Section 3] that one can make the invariant under the Liouville flow outside

a compact subset (which we take to be U) without affecting Floer homology. Therefore they are admissible of type (iv) and (with Lemma 4.4 in mind) one can apply Theorem 1.1. The maps $\mu_{I,J}$ between the adjacent vertices of the hypercube, which are given by counting quilted triangles, were used in [18] to define homomorphism $Kh_s(S)$ corresponding to elementary cobordisms S between links (and more generally tangles). To identify the E_1 page we use the following result which is proved in Theorem 5.6 in [18]. (See also Prop. 5.12 therein.)

Proposition 7.2. *Let K, K' be two flat unlinks in the plane, related by an elementary cobordism S . Then, with coefficients in $\mathbb{Z}/2$, one has a commutative diagram*

$$(29) \quad \begin{array}{ccc} Kh_s(K) & \xrightarrow{Kh_s(S)} & Kh_s(K') \\ \downarrow & & \downarrow \\ Kh(K) & \xrightarrow{Kh(S)} & Kh(K') \end{array}$$

where the vertical arrows are isomorphisms.

Therefore we obtain the following.

Proposition 7.3. *For each link K there is a spectral sequence whose E_2 term is the Khovanov homology with coefficients in $\mathbb{Z}/2$ of K and converges to $Kh_s(K)$.*

7.2. Heegaard-Floer homology. In this section we prove Theorem 1.3. Let Σ be a surface of genus g and equip $Sym^g \Sigma$ with the Kähler form in the cohomology class P_Σ from Example 3.2. For the coisotropic submanifolds $V_\gamma \subset Sym^g \Sigma$ the inclusion map is injective on the fundamental groups and therefore $\pi_2(Sym^g \Sigma, V_\gamma) = 0$. Since (with notation from Example 3.2) $c_1(Sym^k \Sigma) = (k+1-g)\eta - \theta$ and the integral of θ over spheres is zero, the above Kähler form makes $Sym^g \Sigma$ into a spherically monotone symplectic manifold. The submanifolds V_γ are also balanced; the needed line bundle is given by the anticanonical bundle of $Sym^g \Sigma$. See for example Section 6.1 in [3]. Therefore they are admissible of type (iii).

Let M be a 3-manifold given by gluing a genus g handlebody H to another such handlebody H' by ϕ where ϕ is an element of the mapping class group of $\Sigma = \Sigma_g = \partial H$. Let H and H' be given respectively by attaching disks to circles $\alpha_1, \dots, \alpha_g$ and β_1, \dots, β_g in Σ . It is easy to see that (Σ, α', β) , where $\alpha' = (\phi(\alpha_1), \dots, \phi(\alpha_g))$ and $\beta = (\beta_1, \dots, \beta_g)$, form a Heegaard diagram for M . Let $T_\alpha = \alpha_1 \times \alpha_2 \times \dots \times \alpha_g$ and $T_\beta = \beta_1 \times \beta_2 \times \dots \times \beta_g$.

Lemma 7.4. *If γ is an embedded circle in Σ and the coefficient of θ in P_Σ is positive then $\tau_{V_\gamma}(T_\alpha)$ is Hamiltonian isotopic to $T'_\alpha = \tau_\gamma \alpha_1 \times \tau_\gamma \alpha_2 \times \dots \times \tau_\gamma \alpha_g$.*

Proof. By a result of Lekili [6, 3.4.1], T_α is Hamiltonian isotopic to $V_{\alpha_1} \circ \dots \circ V_{\alpha_g}$. One way to prove the lemma is to note that by the naturality of vanishing cycle construction (and the isotopy of monodromy with fibered Dehn twist along the vanishing cycle) we have

$$(30) \quad \tau_{V_\gamma}(V_\alpha) \cong V_{\tau_\gamma \alpha}.$$

Alternatively we can consider a Lefschetz fibration p over the unit disk in which the curve γ gets pinched to a point. Let π denote the relative Hilbert scheme of the fibration p . Let $\alpha_i^t \subset p^{-1}(e^{it})$ for $1 \leq i \leq g$ be a smooth family so that $\alpha_i^0 = \alpha_i$ and $\alpha_i^{2\pi} = \tau_\gamma \alpha_i$. We can construct a Heegaard torus $T_{\alpha,t} := V_{\alpha_1^t} \circ \dots \circ V_{\alpha_g^t} \subset \pi^{-1}(e^{it})$. By pulling back these tori into $\pi^{-1}(1)$ using parallel transport maps, we get a Lagrangian isotopy between T_α and $\kappa^{-1}(T'_\alpha)$ where κ is the monodromy of π which

by [15, Thm 2.19] is Hamiltonian isotopic to fibered Dehn twist along V_γ . Since an isotopy through balanced Lagrangians is always given by a Hamiltonian isotopy, the result follows. \square

Now let $\phi = \tau_{\gamma_k} \circ \cdots \circ \tau_{\gamma_1}$ be an expression of ϕ as a composition of (classical) Dehn twists along a number of curves $\gamma_1, \dots, \gamma_k$ in Σ . Set $\phi_* = \tau_{V_{\gamma_k}} \circ \cdots \circ \tau_{V_{\gamma_1}}$. It follows from Lemma 7.4 that $T_{\alpha'}$ is Hamiltonian isotopic to $\phi_*(T_\alpha)$. As in [17] the Heegaard tori are Lagrangian for our choice of the symplectic form. Let h be a Hamiltonian diffeomorphism which makes the Heegaard tori admissible. Therefore the hat version of the Heegaard-Floer homology of M is given by

$$(31) \quad \widehat{HF}(M) = HF(T_\beta, h \circ \phi_*(T_\alpha))$$

where the Floer homology is taken in $Sym^g(\Sigma) \setminus (z \times Sym^{g-1}\Sigma)$. Let n_z denote the intersection number with the hypersurface $z \times Sym^{g-1}\Sigma$. Note that a sequence of holomorphic curves with $n_z = 0$ will not converge to a curve with $n_z \neq 0$ because of the additivity of n_z and the fact that $n_z \geq 0$ for holomorphic curves. Theorem 1.3 now follows from Theorem 6.1.

7.3. The spectral sequence of a branched double cover. In this section we show that the spectral sequence for branched double covers of links is a special case of the spectral sequence of Theorem 1.3. Let K be a link in S^3 given as the plat closure of a braid b on $2m$ strands. We add two auxiliary strands to b which are not linked with each other or with other strands. We denote the resulting $2m + 2$ braid by b' . Let $B \subset S^3$ be a ball such that $B \cap K = b'$ and $S = \partial B$ intersects K transversely in $4m + 4$ points. The branched double cover of $S^3 \setminus B$ consists of two handlebodies whose boundary is a surface Σ of genus m . It is given by attaching disks to the β curves in Figure 8.

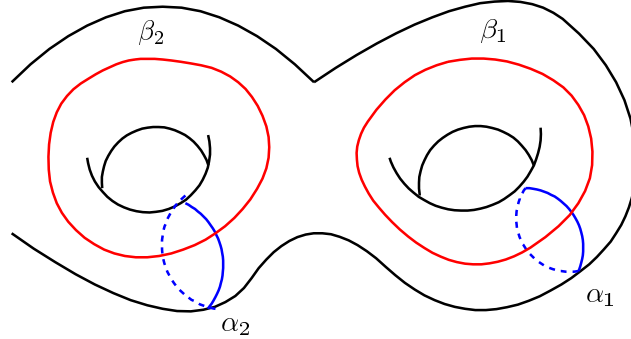


FIGURE 8. Special α and β curves

The double cover of B branched over b' is the mapping cylinder of an element ϕ of the mapping class group of Σ which is given as follows. Let $b = \sigma_{k_N}^{\epsilon_N} \cdots \sigma_{k_1}^{\epsilon_1}$ be an expression of b in terms of braid generators. Let β_1, \dots, β_m be as in Figure 8 and $\gamma_1, \dots, \gamma_{m-1}$ be curves in Σ such that γ_i meets each one of β_i and β_{i+1} in exactly one point and does not intersect other β curves. Let δ_{2i-1} be equal to β_i for $i = 1, \dots, m$ and δ_{2i} equal to γ_i for $i = 1, \dots, m-1$. Then

$$\phi = \tau_{\delta_{k_N}}^{-\epsilon_N} \circ \cdots \circ \tau_{\delta_{k_1}}^{-\epsilon_1}.$$

This is because the conventions for positive braids and positive Dehn twists are opposite of each other.

Lemma 7.5 (Lekili, Perutz). *The Perutz correspondences associated to the curves β_i, γ_i satisfy the following relations where \cong denotes Hamiltonian isotopy.*

- (i) $V_{\beta_i} \circ V_{\gamma_{i+1}}^t \cong \Delta_{\text{Sym}^{m-1}\Sigma_{m-1}}$
- (ii) $V_{\beta_1} \circ \cdots \circ V_{\beta_m} \cong \beta_1 \times \cdots \times \beta_m$

Both statements follow from work in progress of Lekili and Perutz [7] and (ii) is a special case of a theorem proved by Lekili in his thesis [6, 3.4.1] for a general set of nonintersecting curves. Note that (i) is obvious in case $m = 1$. The proof for the general case uses a delicate degeneration argument.

For $I \in \{0, 1\}^I$ let \mathbf{V}^I denote the generalized Lagrangian correspondence $(T_\alpha, V_{\delta_N}^{I_N}, \dots, V_{\delta_1}^{I_1}, T_\alpha)$. The above lemma together with the obvious relation $\tau_{V_\delta} \circ \tau_{V_\delta}^{-1} = id$ are enough to conclude that each \mathbf{V}^I is equivalent in the symplectic category to a generalized correspondence of the form

$$(32) \quad (\beta_1 \times \cdots \times \beta_k, \beta_1 \times \cdots \times \beta_k)$$

in $\text{Sym}^k \Sigma_k$ with $k \leq m$. Basically the relations of Lemma 7.5 imply that the correspondences V_{δ_i} satisfy the same commutation relations as the flat tangles they are assigned to. Let A be the ungraded Khovanov's algebra H^1 over $\mathbb{Z}/2$ i.e. it is generated over $\mathbb{Z}/2$ by $1, X$ with relations $X^2 = 0, 11 = 1, 1X = X$ and with no grading. After Hamiltonian isotoping the above correspondence to make it balanced (which is the same as admissibility of the corresponding Heegaard diagram in this case), the Lagrangian Floer homology of (32) is, as a vector space, isomorphic to $A^{\otimes k}$. This is because β_i and a Hamiltonian isotoped copy of it intersect at two points which are both cocycles.

Let $Kh_{k,k+1} : A^{\otimes k} \rightarrow A^{\otimes k+1}$ and $Kh_{k,k-1} : A^{\otimes k} \rightarrow A^{\otimes k-1}$ be given by Khovanov's TQFT. More specifically $Kh_{2,1}$ is the multiplication map on A and $Kh_{1,2}$ sends 1 to $1 \otimes X + X \otimes 1$ and X to $X \otimes X$. Our notation is misleading because it does not specify on which factor of $A^{\otimes k}$ the maps are acting but this will be clear from the context.

Proposition 7.6. *If J is an immediate successor for I then we have a commutative diagram where the vertical arrows are isomorphisms and $k' = k \pm 1$.*

$$(33) \quad \begin{array}{ccc} HF(\mathbf{V}^I) & \xrightarrow{\mu_{I,J}} & HF(\mathbf{V}^J) \\ \downarrow & & \downarrow \\ A^{\otimes k} & \xrightarrow{Kh_{k,k'}} & A^{\otimes k'} \end{array}$$

Proof. Using the relations of Lemma 7.5 (together with $\tau_V \tau_V^{-1} = id$) the Lagrangian correspondences \mathbf{V}^I and \mathbf{V}^J become equivalent to correspondences either of the form $\mathbf{W} = (T'_\beta, V_{\beta_j}, V_{\beta_j}^t, T'_\beta)$ for some j or $\mathbf{U} = (T'_\beta, T'_\beta)$ where $T'_\beta = \beta_1 \times \cdots \times \beta_k$ for some $k \leq m$. The set of equivalences that give \mathbf{U} and \mathbf{W} from \mathbf{V}^I and \mathbf{V}^J correspond to a sequence of strip shrinking (and/or unshrinking) in the quilt $Q_{I,J}$ which results in a quilted pair of pants $Q'_{I,J}$ which is labelled by \mathbf{U} and \mathbf{W} as in Figure 9. We apply Theorem 6.4 to $Q_{I,J}$ and $Q'_{I,J}$ which reduces the problem to computing the maps induced by $Q'_{I,J}$.

A complex structure (instead of a family) is enough to achieve transversality for transversely intersecting curves on surfaces as in [22, 13b]. We show that the moduli space of pseudoholomorphic maps of $Q'_{I,J}$, given by complex structures on $\text{Sym}^k \Sigma_k$ and $\text{Sym}^{k'} \Sigma_{k'}$ induced from those on

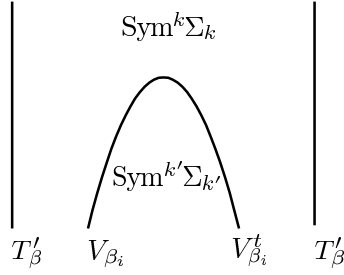


FIGURE 9. The quilted triangle $Q'_{I,J}$ used in the proof of Prop. 7.6. Here $k' = k - 1$.

$\Sigma_k \Sigma_{k'}$, can be described in terms of triangles in Σ_k and therefore one can deduce the transversality of the former moduli space. With the same notation as in Prop. 5.4, the moduli space of holomorphic maps of $Q'_{I,J}$ is the same as the moduli space of pseudoholomorphic triangles u in $Sym^k\Sigma_k$ with boundary conditions given by Lagrangians $T'_\beta, h(T'_\beta)$ (h is a Hamiltonian isotopy) and the coisotropic V_{β_j} for which $\pi \circ u|_\gamma$ can be filled with a pseudoholomorphic disk v in $Sym^{k'}\Sigma_{k'}$.

With the choice of complex structures as above, u is given by a map u' of a suitable branched cover of the disk into Σ_k itself. To avoid the basepoint this map u' has to consist of k holomorphic maps from the disk into Σ_k . Therefore u is determined by k holomorphic maps u_1, \dots, u_k from the disk into Σ_k . To avoid the basepoint, (after possible renumbering) u_i has to send the boundary components of $Q'_{I,J}$ to β_i and a Hamiltonian isotopic copy of it. For the same reason there is a unique $1 \leq j \leq k$ such that if $i \neq j$ the image of u_i does not intersect β_j and therefore, the (interior) seam condition V_{β_j} does not impose any further restriction on u_i ⁵ but it implies that u_j sends the seam to a third isotopic copy of β_j .

Note that for such a u the disk v always exists because $\pi \circ u|_\gamma$ consists of k' nullhomotopic circles in $\Sigma_{k'}$. (This in particular implies that each u_i for $i \neq j$ is a pseudoholomorphic strip of Maslov index zero and is therefore constant.) Therefore the computation is reduced to the genus one case and there are two cases to consider: $k = 2, k' = 1$ and $k = 1, k' = 2$. (Note that the symplectic form induces on a punctured torus is exact and therefore the balanced assumption on the Lagrangians is reduced to exactness.) For the first case one can see by direct inspection of the genus one Heegaard diagram that the pair of pants map acts on the generators in the same way as the multiplication in A . This can also be seen from the fact that $\widehat{HF}(S^1 \times S^2)$ is freely generated, over $H^*(S^1)$, by one element. The second case is also proved by inspecting the Heegaard diagram. See Figure 10. □

Now the Theorem 1.4 follows from Theorem 1.3 using Lemma 7.6 and Prop. 6.5. Note that the extra two strands we added to b adds an unlinked unknot to K and we have $\Sigma(K \cup \bigcirc) = \Sigma(K) \# S^1 \times S^2$.

REFERENCES

- [1] Seong-Gi Ahn. Boundary identity principle for pseudo-holomorphic curves. *J. Math. Kyoto Univ.*, 45(2):421–426, 2005.

⁵This is because if $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_k$ are points on Σ_k away from β_j and $z_j \in \beta_j$ then $(z_1, \dots, z_k) \in V_{\beta_j}$.

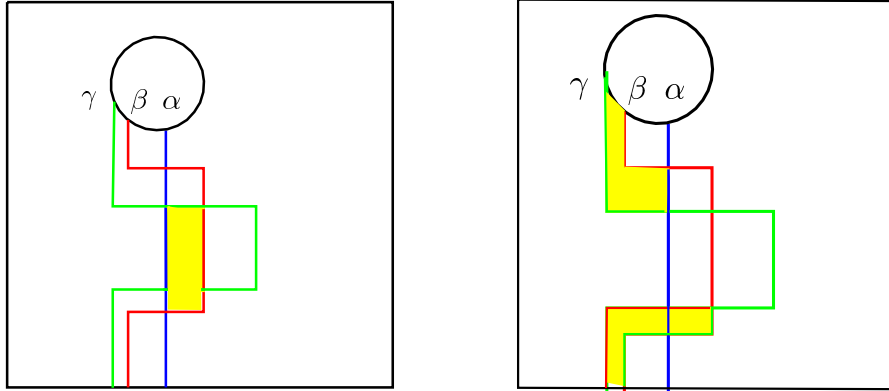


FIGURE 10. The two holomorphic triangles (with one incoming and two outgoing ends) which tell us that θ^+ is sent to $\theta^+ \otimes \theta^- + \theta^- \otimes \theta^+$ by $\mu_{(0),(1)}$ (for a negative twist).

- [2] Anton Alekseev, Anton Malkin, and Eckhard Meinrenken. Lie group valued moment maps. *J. Differential Geom.*, 48(3):445–495, 1998.
- [3] Denis Auroux. Fukaya categories of symmetric products and bordered Heegaard-Floer homology. *J. Gökova Geom. Topol. GGT*, 4:1–54, 2010.
- [4] Paul Biran and Michael Khanevsky. A Floer-Gysin exact sequence for Lagrangian submanifolds. *arXiv:1101.0946*.
- [5] Victor Guillemin and Shlomo Sternberg. *Symplectic techniques in physics*. Cambridge University Press, Cambridge, second edition, 1990.
- [6] Yanki Lekili. *Broken Lefschetz fibrations, Lagrangian matching invariants and Ozsvath-Szabo invariants*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Massachusetts Institute of Technology.
- [7] Yanki Lekili and Timothy Perutz. Lagrangian correspondences and invariants of three manifolds with boundary. *in preparation*.
- [8] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston. Bordered floer homology and the branched double cover I. *arXiv:1011.0499*.
- [9] Ciprian Manolescu. Link homology theories from symplectic geometry. *Adv. Math.*, 211(1):363–416, 2007.
- [10] S. Ma'u, K. Wehrheim, and Woodward C. A_∞ functors for Lagrangian correspondences. *preprint available at math.rutgers.edu/~ctw*.
- [11] Sikimeti Mau. Gluing pseudoholomorphic quilted disks. *arXiv:0909.3339*.
- [12] Yong-Geun Oh. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I. *Comm. Pure Appl. Math.*, 46(7):949–993, 1993.
- [13] Yong-Geun Oh. Floer homology and its continuity for non-compact Lagrangian submanifolds. *Turkish J. Math.*, 25(1):103–124, 2001.
- [14] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.
- [15] Tim Perutz. Lagrangian matching invariants for fibred four-manifolds. I. *Geom. Topol.*, 11:759–828, 2007.
- [16] Timothy Perutz. A symplectic gysin sequence. *arXiv:0807.1863*.
- [17] Timothy Perutz. Hamiltonian handleslides for Heegaard Floer homology. In *Proceedings of Gökova Geometry-Topology Conference 2007*, pages 15–35. Gökova Geometry/Topology Conference (GGT), Gökova, 2008.
- [18] Reza Rezazadegan. Pseudoholomorphic quilts and Khovanov homology. *arXiv:0912.0669*.
- [19] Reza Rezazadegan. Seidel-Smith cohomology for tangles. *Selecta Mathematica New Series*, 15:487–518, 2009.
- [20] Paul Seidel. Graded Lagrangian submanifolds. *Bull. Soc. Math. France*, 128(1):103–149, 2000.
- [21] Paul Seidel. A long exact sequence for symplectic Floer cohomology. *Topology*, 42(5):1003–1063, 2003.
- [22] Paul Seidel. *Fukaya categories and Picard-Lipschitz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

- [23] Paul Seidel and Ivan Smith. A link invariant from the symplectic geometry of nilpotent slices. *Duke Math. J.*, 134(3):453–514, 2006.
- [24] Liam Watson. A remark on Khovanov homology and two-fold branched covers. *Pacific J. Math.*, 245(2):373–380, 2010.
- [25] K. Wehrheim and Woodward C. Exact triangle for fibred Dehn twists. *preprint available at math.rutgers.edu/~ctw*.
- [26] Katerin Wehrheim and Christopher Woodward. Floer cohomology and geometric composition of lagrangian correspondences. *arXiv 09051.1369*.
- [27] Katrin Wehrheim and Chris T. Woodward. Quilted Floer cohomology. *Geom. Topol.*, 14(2):833–902, 2010.
- [28] Katrin Wehrheim and Christopher Woodward. Pseudoholomorphic quilts. *ArXiv:0905.1369*.
- [29] Katrin Wehrheim and Christopher T. Woodward. Floer field theory for tangles. *preprint available at math.rutgers.edu/~ctw*.

Laboratoire de Mathématiques Jean Leray, Université de Nantes
rezazadegan-r@univ-nantes.fr